GENERALIZED CONVEXIETY OF THE FUNCTIONS

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1 Introduction

It is well known that Kolmogorov mean is a function $A_\varphi^n(x_1,\dots,x_n)=\varphi^{-1}\left(\frac{\varphi(x_1)+\cdots+\varphi(x_n)}{n}\right)$ for any $x_1,\dots,x_n\in\mathbb{R}$ from domain of a continuous strictly monotonic function φ. In case $\varphi(x)=x^r$ for $r\neq 0$ and $\varphi(x)=\ln x$ for $r=0$ we have power mean. We use A_p^n to denote it.

Recall that a function $f(x)$ defined on an interval $I\subset\mathbb{R}$ is called convex on I if $f(\lambda x+(1-\lambda)y)\leq \leq\lambda f(x)+(1-\lambda)f(y)$ for any $x,y\in I$ and any $\lambda\in[0;1]$.

Note that $f(A_1(x,y))\leq A_1(f(x),f(y))$ if $\lambda=\frac{1}{2}$.

Definition. A function $f:\mathbb{R}_+\to\mathbb{R}_+$ is called $A_h^nA_g^n$-convex if $n\in\mathbb{N}$, h and g be continuous strictly monotonic functions defined on $\mathbb{R}_+$ and $A_h^n\big(f(x_1),\dots,f(x_n)\big)\geq \geq f(A_g^n(x_1,\dots,x_n))$ for any $x_1,\dots,x_n\in\mathbb{R}_+$.

The purpose of this research work is to find criteria of $A_h^nA_g^n$-convexity of the functions and study the applications of these functions in solving inequalities.

This problem is well-known for $n=2$. For example, in [1] and [2] some necessary and sufficient conditions of $A_p^2A_q^2$-convexity were obtained.

2 Results

Theorem 1. *Let $n\in\mathbb{N}$, f,g and h be continuous functions defined on $\mathbb{R}_+$ and g,h be strictly monotonic. Then f is $A_h^n\,A_g^n$-convex if and only if $h(f(g^{-1}(x))$ is convex (concave) on $g(\mathbb{R}_+)$ if h is increasing (decreasing).*

From this theorem follow *Theorem 2.4.* [1] where criteria for $A_p^2\,A_q^2$-convexity were obtained for $p,q\in\{-1,0,1\}$ and *Lemma 3* [2] where criteria for $A_p^2\,A_q^2$-convexity for differentiable functions were obtained.

Corollary 1.1. *Let f,g,h be twice differentiable functions, $f:\mathbb{R}_+\to\mathbb{R}_+$, h and g be strictly monotonic functions, $n\in\mathbb{N}$ and $g'(x)\neq 0$, $x\in\mathbb{R}_+$. Then f is $A_h^n\,A_g^n$-convex if and only if $h''\big(f(x)\big)\,[f'(x)]^2+f''(x)\,h'\big(f(x)\big)- -\frac{g''(x)}{g'(x)}h'\big(f(x)\big)\,f'(x)\geq(\leq)\,0,\ x\in\mathbb{R}_+$ if h is increasing(decreasing).*

Corollary 1.2. *Let f be a twice differentiable function, $f:\mathbb{R}_+\to\mathbb{R}_+, n\in\mathbb{N}$, p and $q\in\mathbb{R}_+$ Then f is $A_p^nA_q^n$-convex if and only if $(p-1)x\,[f'(x)]^2+xf''(x)f(x)- -(q-1)f'(x)f(x)\geq 0, x\in\mathbb{R}_+$.*

The following corollary helps in comparing one Kolmogorov mean to another.

Corollary 1.3. *Let h and g be continuous strictly monotonic functions on $\mathbb{R}_+$. If $h(g^{-1}(x))$ is convex (concave) on $g(\mathbb{R}_+)$ and h is increasing (decreasing), then for any positive $x_1,\dots,x_n$ $A_h^n(x_1,\dots,x_n)\geq A_g^n(x_1,\dots,x_n)$.*

Theorem 2. *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a $A_1^2A_0^2$-convex function. Then $\sum_{i=1}^n f(a_ib_{n-i+1})\leq\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n f(a_ib_j)\leq\sum_{i=1}^n f(a_ib_i)$ for any real positive numbers $a_1\leq\cdots\leq a_n$ and $b_1\leq\cdots\leq \leq b_n$.*

Since $f(x)=x$ is a $A_1^2A_0^2$-convex function then as a corollary we have well-known Chebyshev's sum inequality.

Theorem 3. *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be $A_0^2A_0^2$-convex function. Then $\sqrt{\frac{f(a_1)+\cdots+f(a_n)}{n}\cdot\frac{f(b_1)+\cdots+f(b_n)}{n}}\geq\frac{f(\sqrt{a_1b_1})+\cdots+f(\sqrt{a_nb_n})}{n}$.*

Since $f(x)=x^2$ is $A_0^2A_0^2$-convex function then as a corollary we have Cauchy-Schwarz inequality.

Theorem 4. *Let f be a continuous $A_1^2A_0^2$ -convex function, a_1,a_2,a_3,b_1,b_2,b_3 are such non-negative numbers that $A(a_1,a_2,a_3)\succ B(b_1,b_2,b_3)$ Then $\sum_{sym}f(x^{a_1}y^{a_2}z^{a_3})\geq \geq\sum_{sym}f(x^{b_1}y^{b_2}z^{b_3})$ for any real positive numbers x,y,z.*

Since $f(x)=x$ is a $A_1^2A_0^2$-convex function as a corollary we have Muirhead's inequality.

Theorem 5. *Let h and g be continuous monotonic functions defined on $\mathbb{R}_+$ and let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a $A_h^2A_g^2$-convex function, $n\in\mathbb{N}$. Then for any $y,\dots,y_n\in\mathbb{R}_+$*

$$A_h^n(A_h^n\big(f(y_1),f(y_2),\dots,f(y_n)\big),f\left(A_g^n(y_1,y_2,\dots,y_n)\right),\dots,f\left(A_g^n(y_1,y_2,\dots,y_n)\right))\geq$$
$$\geq A_h^n(f(A_g^{n-1}(y_2,y_3,\dots,y_n)),f(A_g^{n-1}(y_1,y_3,\dots,y_n)),\dots,f(A_g^{n-1}(y_1,y_3,\dots,y_n))).$$

As corollary we obtain Popoviciu's inequality in Vasile Cîrtoaje form (Theorem 3a, [3]).

3 Conclusion

In this paper necessary and sufficient conditions of $A_h^nA_g^n$-convexity of f, functional generalizations of Chebyshev's sum inequality, Cauchy-Schwarz inequality and Muirhead's inequality, Popoviciu's inequality were obtained.

The results were obtained by the methods of mathematical and functional analysis.

These results can be useful in nonlinear programming and for such applications as mathematical optimization which has a specialized subsection – convex analysis.

4 References

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[2] Ápád Baricz. Geometrically concave univariate distributions // J. Math. Anal. Appl. – 2010. – V. 363 – P. 182–196.

[3] Darij Grinberg, Generalizations of Popoviciu's inequality // ArXiv.org e-Print archive, arXiv: 0803.2958v1, 20 March 2008.