On two-letter identities in Lie rings

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Introduction

Abstract algebra is a branch of mathematics that studies algebraic structures. Lie algebras theory is one of the most important and developing topic of abstract algebra. A Lie ring is a generalization of a Lie algebra. Lie algebras firstly arose in a study of differential equations in 1880s. These structures occur in different areas of science such as quantum physics, combinatorial group theory and geometry. By definition, Lie ring is an abelian group together with a non-associative operation called ``Lie bracket''. This operation is denoted by [•, •]. By definition a Lie bracket is a $\mathbb{Z}-$bilinear map satisfying two identities [a, a] = 0 and [[a, b], c] + [[c, a], b] + [[b, c], a] = 0. The last identity is called the Jacobi identity. Define by recursion the left normed commutator as follows $[a_1, \dots, a_n] = [[a_1, \dots, a_{n-1}], a_n]$. We can derive the following identity from axioms.

$$[a, b, a, b] = [a, b, b, a]. \quad (1)$$

Set $[a,_0 b] = a$ and define $[a,_k b]$ by recursion $[a,_{k+1} b] = [[a,_k b], b]$. The following generalization of the identity (1) was obtained by R. Mikhailov and S. Ivanov in [1]

$$\left[[a,_{2n}\ b], a\right] = \left[\sum_{i=0}^{n-1}(-1)^i\left[[a,_{2n-1-i}\ b], [a,_i\ b]\right], b\right] \quad (2)$$

for all $n \geq 1$. This identity was used to prove the main result of their work. Our research is devoted to the problem of description of similar identities in more generalized way.

Research methods

- Theoretical proofs. All results were proven in theoretical way using the language of abstract algebra. Some new ways of representation of considered problem were studied.
- Programming. Using obtained results, we created a computer program that can generate very long non trivial identities.

Results

We described all possible identities with two "a" letters and any number of "b" letters. More specifically, there are no identities with odd number of "b" letters and only one with even number of them up to multiplication by a coefficient. Also, we used and described absolutely new approach to the problem of finding two letter identities. This approach relies on combinatorial-geometric interpretation of this problem. We considered commutators as dots on integer plane and transformed algebraic theorems into graphical form. Using this method, we obtained the main result of our work. It is a nontrivial identity with three "a" letters and arbitrary number of "b" letters that is divisible by three. Another important result is a proof that there are no nontrivial identities with odd length of commutators lesser than 9 and the very first identity with odd number of elements in commutators. Besides of that, we described an algorithm that can generate any identities with three "a" letters. Moreover, we obtained some statements about ranks of a free abelian groups that generate such identities.

Conclusion

Description of identities with any number of "a" letters will lead to description of all two letter identities in arbitrary Lie rings and algebras. Such result could be really helpful for researchers in the Lie rings theory. Cases that were studied in our work describe quite big class of identities and can be used in the same topic as they were used in [1]. Finally, our results create a connection between some combinatorial questions and Lie rings theory, providing a straight simplification in some cases.

References

[1] Sergei O. Ivanov, Roman Mikhailov, A finite $\mathbb{Q}-$bad space, arXiv:1708.00282

[2] C. Reutenauer, Free Lie algebras, Oxford University Press, 1993.